Slit domains.

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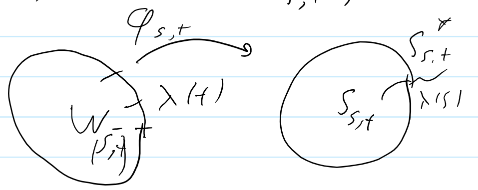
Let $\gamma : [0,+\infty) \to \mathbb{D}$, $\gamma(0) \in \partial\mathbb{D}$, $\gamma(\infty) = 0$. Can be self-touching! Let $\Omega_t :=$ component of $\mathbb{D} \setminus \gamma[0,t]$, containing 0. Let $f_t : \mathbb{D} \to \Omega_t$, $f_t(0) = 0$, $f_t'(0) > 0$. Normalize: $|f_t'(0)| = e^{-t}$: L.C (normalized). By Caratheodory, f_t extends continuously to $\partial\mathbb{D}$. Let $\lambda(t) := f_t^{-1}(\gamma(t))$. As before, $g_t := f_t^{-1}$, $g_t(\gamma(t)) = \lambda(t)$.

Thm. $\dot{g}_t(z) = g_t(z)\dfrac{z+\lambda(t)}{z-\lambda(t)}$. $\lambda(t)$ is called driving function of $\gamma(t)$.

Pf. (g_t) is normalized L.C., so $\dot{g}_t(z) = g_t(z)\, p(g_t(z),t)$, for some p. Remember that

$$p(z,s,t) = \frac{1+e^{s-t}}{1-e^{s-t}} \cdot \frac{z-\varphi_{s,t}(z)}{z+\varphi_{s,t}(z)} = \int \frac{z+\zeta}{z-\zeta}\, d\mu_{s,t}(\zeta),$$

where $\mu_{s,t}(\zeta)$ is supported on the set $\overline{\{|\varphi_{s,t}(\zeta)| \neq 1\}}$, i.e.



$W(s,t) := f_t^{-1}(\gamma[s,t]) = g_t(\gamma[s,t])$.

Let $S_{s,t} := g_s(\gamma[s,t]) = \varphi_{s,t}(W(s,t))$

Use Shwarz reflection to extend $\varphi_{s,t}$ to conformal isomorphism of $\hat{\mathbb{C}} \setminus W(s,t)$ onto $\hat{\mathbb{C}} \setminus (S \cup S^*)$. By Caratheodory, $|S_{s,t}| \to 0$ as $t \downarrow s$.

On the other hand, $|W(s,t)| \to 0$ as $s \uparrow t$ (since $|W(s,t)|$ is the harmonic measure of $\gamma[s,t]$ in Ω_t).

Observe: 1) for s close enough to t, $W(s,t) \subset B(\lambda(t), \varepsilon)$.

2) $\varphi_{s,t} \to id$ uniformly on compact subsets of $\mathbb{D}$. By reflection, on compact subsets of $\hat{\mathbb{C}} \setminus S^1$.

Contour $C(0,2) - C(\lambda(t), \delta)$, let $\delta \to 0$. We see that convergence is uniform on $\partial B(\lambda(t), \varepsilon)$. Thus $\varphi_{s,t}(W(s,t))$ lies inside $B(\lambda(t), \varepsilon)$ for s close to t, so $|\lambda(s) - \lambda(t)| < \varepsilon$ for this s, i.e. $\lim_{s\uparrow t} \lambda(s) = \lambda(t)$.

The same argument gives $W(s,t) \to \lambda(s)$ when $t \downarrow s$ (since $\varphi_{s,t}^{-1} \to id$).

By $W(s,t) = \operatorname{supp} \mu_{s,t}$, thus $\mu_{s,t} \to \delta_{\lambda(t)}$ weakly as $s \to t$.

So $\lim_{s\to t} p_{s,t}(z) = \dfrac{\lambda(t)+z}{\lambda(t)-z}$ always exists!

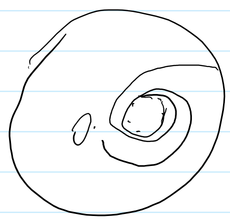
Other direction almost holds!

Thm (Pommerenke) The L.C. (Ω_t) is generated by continuous $\lambda(t)$ iff $\forall T > 0, \varepsilon > 0, \exists \delta > 0 : \forall t \leq T$ $\exists$ crosscut γ of Ω_t separating 0 from $K_{t+\delta} \setminus K_t$, $|\gamma| < \varepsilon$.

The existence of λ is the same as in slit case. Other direction requires some work.

Example (of non curve):

[illegible]



Chordal analogue:

$$g_t(z) = \frac{2}{g_t(z) - \lambda(t)}, \quad \lambda(t) = g_t(\gamma(t)).$$

Bonus: Brownian scaling! Driving function of αK_t is

$$\alpha^{-1} \lambda(\alpha^2 t) \quad \left(\text{since } g_{\alpha K}(z) = \alpha\, g_K\!\left(\tfrac{z}{\alpha}\right)\right).$$